eBay 9/11

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Abstract

This paper analyzes a unique dataset of art auctions on eBay by means of a novel structural estimation approach. Our empirical framework considers the process of arrival of new bidders as well as the distribution of bidders valuations of artworks being auctioned. From a methodological perspective we develop a new econometric approach for estimation of second-price ascending-bid auctions with a stochastic number of bidders and missing data. The proposed approach uses the available information efficiently and avoids problems of selection bias present in existing studies. In an empirical application we use this econometric framework to quantify the economic impact of the September 11 shock in this particular market. Our results indicate that the events of 9/11 had a small, and short lived, negative impact.

JEL CLASSIFICATION: C51, C72, D44, L11, L14.

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1 Introduction

The standard model of a competitive market in which sellers post prices and buyers make purchasing decisions does not always represent the most efficient mechanism of exchange. Consider for example the market for original art. Because each work of art is unique, its selling price may be difficult to determine ex-ante. In contrast, auctions relieve the seller of the responsibility of setting a price. Thus, it is not surprising that original art is frequently sold in an auction environment (Ashenfelter and Graddy, 2002 and 2003).

This paper analyzes a unique dataset of real art auctions taking place on eBay, a popular auction site on the Internet. We study the behavior of sellers and potential buyers; we present a novel estimation approach; and we use our empirical framework to quantify the economic impact of the 9/11 shock in this particular market. Our research contributes to the growing literature on structural estimation of second-price ascending auctions while paying special attention to the specific features of eBay auctions. Our research also contributes to the growing literature on online markets (Lucking-Reiley, 2000).

Auction research in the econometrics field has been conducted using either a reduced form approach or a structural approach.[2] In the reduced-form approach, the theory is used only as a guide for the econometric specification. Reduced form studies are usually concerned with the testing of theoretical predictions and the analysis of bidding behavior. In particular, some researchers have used auction data from eBay to analyze the importance of a

[2] Recent surveys include Hendricks and Paarsch (1995), Perrigne and Vuong (1999), and Hendricks and Porter (2000).

seller's reputation (Melnik and Alm 2002, Resnick and Zeckhauser 2002). Other researchers, also using data on Internet auctions, analyze the effect of specific auction rules on bidding behavior (Roth and Ockenfels, 2002). A significant body of work has emerged that uses the structural approach, specially for first-price auctions (see the survey by Perrigne and Vuong, 1999), but also for second-price ascending auctions and other types of auction formats. The structural approach incorporates the restrictions implied by the theory explicitly into the empirical work. The econometric specification is then fully consistent with the underlying theoretical model. Researchers have used the structural approach to provide estimates of optimal reserve prices (Paarsch 1997), to analyze the effects of bidder collusion (Baldwin, Marshall and Richard, 1997), to measure the extent of the winner's curse and simulate seller revenue under different reserve prices (Bajari and Hortacsu 2003), or to analyze the effects of congestion in an online auction market (Canals-Cerda, 2005), to name a few.

Most of the existing structural econometric models of second-price ascending auctions, the focus of this paper, have been inspired by the 'button auction' model of Milgrom and Weber (1982). The second-price ascending auction model described in their paper has a unique dominant-strategy equilibrium in which it is a best strategy for each bidder to submit a bid equal to her valuation of the object being auctioned. A prevalent approach, proposed by Donald and Paarsch (1996), assumes that the highest bid of each one of the auction participants, except the winner of the auction, equal their true valuation (see also Paarsch 1997; Hong and Shum 2003; Bajari and Hortacsu 2003). Alternatively, some papers require only that the winning bid equals the valuation of the second-highest bidder (Paarsch 1992;

Baldwin et al. 1997; Haile 2001). In contrast, in a remarkable paper, Haile and Tamer (2003) use information on the highest bid from each bidder, but do not require any bid to equal the bidder's true valuation of the object being auctioned. One of the findings of their paper is that models which use only the available information on winning bids appear to be much more robust to variations in auction format than models that equalize the highest bids from all bidders (except the highest one) to the bidders' true valuation. The approach developed in this paper is similar to Haile and Tamer (2003) in that we also assume that bidders never bid above their true valuation. Unlike Haile and Tamer we assume that the second highest bid equals the bidder's true valuation. In the paper we argue that this is not a strong additional assumption, at least for eBay auctions. Also, unlike Haile and Tamer, in our paper we allow for an unknown and stochastic number of bidders.

Two main features distinguish eBay auctions from the 'button auction' model: the potential number of bidders is unknown, and not all bidders necessarily bid their true valuation. In particular, it is often the case that a potential buyer does not have a chance to place a bid, or does not have a chance to place her highest bid . This can happen if, at the time of bidding, the bidder's valuation is below the current minimum acceptable bid. This is not an unlikely event, especially close to the end of the auction. Only the two bidders with the two highest valuations are guaranteed the opportunity to bid their true valuation at any point during the auction (Athey and Haile, 2005).

Like us, Hasker, Gonzalez and Sickles (2001) and Song (2004), also use eBay data and a structural framework.[3] Both papers deal with the lack of exact information about the number of potential bidders in different ways. Song (2004) presents a valuable identification result: the distribution of bidder's valuation is identified, under the assumption that the bid of the second and third highest bidders equals their valuation. Song also proposes an estimation approach derived from the identification result. Hasker, Gonzalez and Sickles (2001) estimate the number of potential bidders from information on the winning bids and under the assumption that the number of potential bidders is constant across auctions of the same length.[4] Their estimation strategy is based on the simulated non-linear least squares estimation approach popularized by Laffont, Ossard and Vuong (1995).

The key estimation assumption in Hasker, Gonzalez and Sickles (2001), that the winning bid equals the valuation of the second-highest bidder, only holds for auctions with two or more active bidders. Also, the assumption that the number of potential bidders is constant across auctions of the same length is only plausible for homogeneous auctions. Similarly, the identification result in Song (2004) requires at least the presence of three active bidders and that the second and third highest valuations are observed in the data, an assumption that is unlikely to hold for eBay auctions. Therefore, these authors are required to restrict their samples to auctions with at least two, or three, bidders, respectively, which can result in sample selection bias. In fact, a significant proportion of auctions on eBay end up without any bids. These auctions will be excluded from the final dataset used in their estimation.

[3] Bajari and Hortacsu (2003) also make use of data from eBay but they consider a common value auction model and adopt a Bayesian methodology. In contrast, in this paper we consider the independent private value model, like Hasker, Gonzalez and Sickles (2001), Song (2004) and Haile and Tamer (2003).

[4] In particular, their predicted number of bidders for 7 days auctions is 27.

In particular, about 40% of the auctions in our data have no bidders, 60% have less than two bidders and 72% less than three bidders.[5] Both papers measure the effect of a seller's good/bad reputation on the winning bid, and both papers conclude that having a bad reputation has a negative effect on the final selling price. Thus, if a seller's reputation also affects a potential buyer's decision to place a bid, a possibility that is ignored by both papers, their results will be biased in two different ways. First, there will be selection bias due to the fact that their econometric approach forces them to disregard auctions with a small number of bidders, an event that is most likely correlated with the seller having a bad reputation. Second, there will be bias from assuming that the process of arrival of new bidders is not a function of the seller's reputation.

In our opinion, a fully structural econometric model of eBay auctions should take into account the process of bidders' arrival as well as the bidders' valuations. Unlike previous studies, our estimation approach makes use of information on the time of arrival of bids, which is readily available on eBay, along with information on bids from all auction participants. To our knowledge this is the first paper to do that. In addition, our estimation approach takes into account all available auctions, irrespective of the number of bidders, and thus avoids problems of selection bias.

The paper proceeds as follows. In Section two, we describe the data to be used in the analysis. In section three we present a structural model of auctions and describe the econometric methodology. In Section four we apply our methodology to the analysis of the

[5] In the data used by Song, approximately fifty percent of auctions did not sell and less than ten percent of auctions had more than two bidders.

effects of the terrorist attacks of 9/11 in the market of art by self representing artists on eBay. Section fifth concludes. In addition two appendixes are included at the end of the paper. There is a lack of identification results in the empirical literature of auctions with an unknown number of bidders (Athey and Haile, 2005). The first appendix presents several identification results, with the more general setting allowing for the presence of unobserved auction specific heterogeneity, the setting considered in our empirical application. The second appendix presents theoretical results relevant for estimation.

2 The Market and the Data

A. The Market.

The most familiar auction mechanism on eBay is the second-price ascending auction in which bidders compete for an item over a certain time period raising their bids up to their maximum willingness to pay. Sellers are free to offer objects for auction at any time by posting a description of the item being auctioned, including pictures. Sellers can choose the auction length from four possible alternatives: three, five, seven or ten days. Sellers can also choose other auction characteristics, like a reserve price, at a fee. Buyers are able to browse among thousands of auctions posted every day. Auctions are organized by categories, and subcategories, which simplifies the buyer's search. Buyers are also able to use a powerful search engine. In addition, the search/category output can be ordered according to price, time listed and time left for the auction. Potential bidders can also search among completed auctions. This is a useful feature because it allows bidders to learn about the range of prices that a particular seller has been able to obtain in the past, as well as how much competition the auctions usually generate. In addition, bidders who may be influenced by the choices of other bidders can learn much more from the outcomes of a large number of completed auctions than from a particular auction in progress. Buyers and sellers are also able to communicate anonymously by email.

Bidders participating in an auction can submit a proxy bid at any time before the end of the auction. A proxy bid represents the maximum amount a bidder is willing to pay at the time of bidding. Proxy bids can be revised (increased) at any time prior to the end of the auction. As long as a bidder's proxy bid is higher than the second highest bid she will remain the highest bidder, at a price equal to the second highest bid plus a small increment. The auction price will increase every time the second highest bid increases. Bidders can also use "snipping" programs which allow them to place a predetermined bid amount at a chosen time before the end of the auction. Bids can also be retracted, although this does not usually happen. Only bids above the existing second highest bid, plus an increment, are acceptable. The highest bidder at the end of the auction wins the item at a price equal to the second highest bid, plus a small increment.[6]

B. The Data.

In July 2001 we began collecting auction data from a group of artists, self-denominated EBSQ, who sell their own artwork through eBay.[7] Although this group did not necessarily

[6] Descriptions of eBay auctions can be found in Song (2004), Bajari and Hortaçsu (2003, 2004) and Lucking-Reiley et al. (2000).

[7] EBSQ is an abbreviation for "e-basquiat" after the artist Jean-Michel Basquiat who was admired by the group's founding artist. The artists on this group organize online activities, like art contests, and receive discounts at online art stores. For more information on EBSQ, visit http://www.ebsqart.com/.

include all self-representing artists, to our knowledge this was the only group of its kind at the time. In most cases the item for sale was an original work of art, most often a painting but other forms of art like collages, ceramic tiles, or sculptures were also offered. Some artists also sold reproductions of their original works of art. Whenever feasible, we collected data using computer programs. In cases where this was not possible, a specific database was used to aid in the collection of data and to minimize errors. For each auction, we collected four different types of information: item characteristics, auction characteristics, bidding history, and artist reputation. The first category describes characteristics specific to the object being auctioned. Information on the height, width, style (abstract, pop, whimsical, etc.), medium (acrylic, oil, etc.) and ground (stretch canvas, paper, wood, etc.) of the painting was collected. Auction characteristics include the length of the auction, the opening bid, the shipping and handling fees, the eBay category in which the object is being listed, and whether the auction had a reserve price or a 'buy it now' price option.[8] The bidding history lists all the bidders who submitted an acceptable bid at any given time. The bidding history becomes public information at the end of the auction, except for the highest bid which is not reported. At the end of each transaction the buyer and the seller have a chance to rate their level of satisfaction with the transaction by choosing one of three options: positive (1), neutral (0), negative (-1). Thus, we also collect information on the type of feedback received by the seller in previous transactions, and use this information to define two measures

[8] Before any bids have been submitted, a potential buyer can terminate the auction and purchase the item being auctioned by paying a 'buy it now' price set by the seller. After the first bid has been submitted this option disappears.

of artists' reputation: a reputation index representing the percentage of positive feedback received from buyers, which is exactly the same as the reputation index used by eBay; and another measure of artists' reputation equal to the total number of unique buyers. This second measure of artist's reputation is used to proxy the effect of having a large customer base as well as the 'word of mouth' effect.

The data collected consists of all the auctions associated with EBSQ from the third week of July, 2001, to the second week of November 2001. This includes approximately ten thousand auctions of artworks by self-representing artists on eBay. The selling price of the artwork auctioned ranges from less than one dollar to hundreds of dollars, or even thousands of dollars in a small percentage of auctions. Between August 11, 2004 and December 11, 2004 we conducted a second round of data collection on a subgroup from the original group of artists consisting of those artists who posted auctions regularly during the original data collection period. In this study we use data on original paintings only for this subgroup of artists for the years the 2001 and 2004. Also, we do not include in our analysis approximately 2% of the auctions in which the 'buy it now' option was used.

Tables 1 to 5 present descriptive statistics. As indicated in table 1, auctions with zero bidders represent about forty percent of all auctions, but auctions with two, three or four bidders are not uncommon. Eight percent of the auctions in 2001, and sixteen percent in 2004, have more than four active bidders. Multiple bids from single bidders are not uncommon. Artists can choose the auction's length to be three, five, seven or ten days. As indicated in table 2, close to ninety percent of auctions last seven to ten days. Table

3 presents the distribution of paintings across artists. Some artists were more prolific than others over the period of data collection. In particular, our data includes thirty paintings or more for approximately seventy percent of the artist and anywhere between fifteen and thirty paintings for the remaining artists. Table 4 presents information about the timing of arrival of winning bids. Some authors have interpreted late bidding as a sign of complex strategic behavior on the part of the bidders (Roth and Ockenfels, 2002). In our data only a small percentage of winning bids arrive within the last five seconds, one percent in 2001 and three percent in 2004. Bidders wanting to bid close to the end of the auction can use bidding services provided by companies like ezsniper.com, esnipe.com, or auctionwatch.com, which make the process of late bidding very reliable.[9] These services are either very inexpensive or free. The increase in late bidding observed in 2004 may be related to an improvement in the reliability of this type of services. Finally, table 5 presents descriptive statistics for relevant variables. In the econometric analysis shipping costs are added to the value of bids. Other relevant variables not included in this table are style (abstract, contemporary, ...), medium (acrylic, pen and ink,...) and ground (stretch canvas, canvas panel,...), descriptive statistics for these additional variables are available from the authors. The explanatory power of these variables is much higher than what is common in many areas of economics. In particular, a simple log-linear regression of transaction prices for sold items as a function of explanatory variables results in an r-square value of 0.72 when artists' specific fixed effects

[9] For example, in www.auctionsniper.com we can read: "We recommend lead times between 5 and 10 seconds, although we've seen times as low as 2 and 3 seconds work just fine. Currently, Auction Sniper successfully places 99.9% of all snipes with 5 second lead times." EBay neither encourages nor opposes the use of snipping strategies. They point out that buyers can always protect themselves against this type of bidding behavior by using proxy bidding and bidding their maximum willingness to pay early on.

are not included, and 0.78 when they are included.

Several characteristics of this data make it unique. First, the data collected comprises all eBay market activity of a specific group of sellers for a long period of time, while the data collected by other researchers usually represents only a narrow snapshot of market activity. Second, by nature the intrinsic value of artwork is uncertain, especially in the case of less well-known artists. In contrast, much of the data collected by other researchers refers to items for which their market value can be determined with accuracy, like coins, stamps or computers, which reduces the value of auctions as a selling, price-finding, mechanism.

3 A Structural Model of eBay Auctions.

We model eBay auctions as an IPV, ascending-bid, second-price auction (Klemperer, 1999), subject to some specific rules. The seller of the object being auctioned sets the duration of the auction, T, and the starting value, s_0.[10] Each potential bidder, j, assigns a value, v_j, to the object being auctioned, with each value, v_j, being a random realization from a distribution $F(v)$. Bidders know and care only about their own valuation. Bids are submitted electronically at any time t within the $[0,T]$ time interval. At each time t the price of the auction is set at the current second highest bid, say $s(t)$, and the current, or active, bidders are public information. New bids arrive sequentially at any time during the $[0,T]$ interval. Any new bid has to surpass $s(t)$ by a minimum increment in order to

[10] The seller has also the option of adding a hidden reservation price and a "buy it now" price. The "buy it now" option allows a potential buyer to terminate the auction if she is willing to pay the price. However, this option disappears after the first bid is placed.

be recorded.[11] In what follows we avoid mention of this minimum increment, except when absolutely necessary, to avoid unnecessary notation.

A. The Bidders' Arrival Process:

We only model the timing of the first bid by any new bidder. The process we have in mind is one in which at the time of her first bid the bidder incurs a certain cost necessary in order to learn about the product, by reading the product description, looking at the available pictures, and perhaps even emailing the seller with any questions that the potential buyer may have. After the first bid, we assume that the bidder can continue bidding as many times as she wants at no additional cost.

Define the potential number of bidders as the number of bidders that would have chosen to bid in an otherwise identical sealed bid auction in which all bids are acceptable. An important feature to consider when analyzing this type of auction is that when a new bidder arrives at time t with valuation below $s(t)$ this bidder will be unable to bid because the auction price at the time is higher than her valuation. Thus, when studying auctions on eBay we need to be aware of the following facts: the number of potential bidders may be higher than the observed number of bidders; the arrival time of bids and their magnitudes matters; and some potential bids will not be realized. Also, the number of potential bidders is stochastic, a function of the auction characteristics, and unobserved in most cases.

We model the arrival of new bidders to an auction in a way similar to the arrival of job offers in a structural job search model (Flinn and Heckman, 1982), and also similar to

[11] In Ebay the value of the minimum increment varies with s_0. The minimum increment is \$0.05 for bids under \$1.00 dollar and increases up \$100.00 for bids above \$5000.00.

how previous authors have modeled this arrival process in an auction environment (Wang, 1993). Denote the instantaneous probability of arrival of new potential bidders at any time $t \in [0, T]$ as λ. The instantaneous probability of arrival of new bidders is the product of the arrival rate of potential bidders, λ, and the probability that the existing second highest bid, say $s(t)$, is below the new bidder's valuation, or equal to $\bar{F}(s(t)) = 1 - F(s(t))$. Thus, the hazard function for the arrival of new bidders is

$$\alpha(t) = \lambda \bar{F}(s(t)). \tag{1}$$

This also implies that the hazard function for the arrival of potential, unrealized, bids is $\lambda F(s(t))$. Furthermore, implicit in this assumption is the requirement that any potential bidder with valuation above $s(t)$ is equally likely to bid after time t. This assumption will be relaxed in the empirical work.

From an empirical perspective we need to take into account the presence in our data of heterogeneity across auctions. Thus, denote by (X_{ij}, η_{ij}) the vector of relevant characteristics describing object j being auctioned by artist i, with X_{ij} representing characteristics of the object that are observed by the econometrician and η_{ij} a real valued index summarizing all other object specific characteristics that are unobserved. In this framework, we parametrize the instantaneous probability of arrival of new bidders as follows

$$\log \lambda_{ij} = \log \lambda(t) + x_{ij}' \beta + \delta \eta_{ij}, \tag{2}$$

This specification includes a baseline hazard $\lambda(t)$ that can take on different values at different points over the duration of the auction, an index function $x'_{ij}\beta$ to account for observed heterogeneity, and an unobserved heterogeneity component η_{ij} which will be treated as a random effect. This specification resembles a negative binomial count model (Cameron and Trivedi, 1998).[12] Identification of equation (2) requires $\delta = 1$, with no restrictions imposed on the variance of η_{ij}. Identification also requires that the index $x'_{ij}\beta$ be included without an intercept term, with no restrictions imposed on $\lambda(t)$.

The likelihood of bidder arrival is not necessarily uniform over the duration of the auction $[0, T]$. For example, impatient bidders may choose to concentrate their attention on auctions close to their ending time. Also, the available search options on eBay make it easier for a potential buyer to find out about auctions just after they have been posted or close to their ending. We address this point in our empirical model by allowing the baseline hazard $\lambda(t)$ to take on different values over the duration of the auction. More precisely, we divide $[0, T]$ into R mutually exclusive intervals, say $\{I_r\}_{r=1}^{R}$, and allow $\lambda(t)$ to take different values across subintervals, and to remain constant within subintervals, or $\log \lambda(t) = \lambda_r$ for $t \in I_r$, $r = 1, ..., R$. This feature of the arrival process has not been described explicitly in equation (2) to avoid excessive notation. No additional restrictions are imposed on λ.

The process just described refers to the arrival of potential bidders. However, the arrival process that we observe in our data includes only actual bidders, that is, these potential

[12] The negative binomial model corrects two important limitations of the Poisson model: the excess zeros problem, more zeros in the data than the model predicts, and the overdispersion problem, mean and variance are not restricted to be the same.

bidders that were able to bid because their valuation of the item being auctioned was above the auction price at the time they were ready to bid. Consistent with equation (1), we can model the arrival hazard for new potential bidders at a particular time t according to the following parameterization, $\alpha_{ij}(t) = \lambda_{ij}\bar{F}(s_{ij}(t))$, with λ_{ij} defined above and $s_{ij}(t)$ representing the minimum acceptable bid at time t. In fact, in our empirical model we consider a more general specification of the form $\alpha_{ij}(t) = \lambda_{ij}K\left(\bar{F}(s_{ij}(t))\right)$,with $K : [0,1] \rightarrow [0,1]$ non-decreasing (the larger $\bar{F}(z)$ the higher the chances of participation). Also we assume that $K(0) = 0$ and $K(1) = 1$. The first equality implies that if the current minimum acceptable bid is such that there are no potential bidders with a high enough valuation to improve upon it, that is if $\bar{F}(s_{ij}(t)) = 0$, then there will be no new bids. The second equality is required for identification due to the multiplicative form of α_{ij}.[13] This specification is equivalent to the one described in equation (1) for $K(z) = z$. However, in general this specification takes into account the possibility that the probability of arrival of new bidders may not be a linear function of $\bar{F}$. For example, this could be the case if potential bidders with higher valuation also have a higher propensity to bid.[14] Any function $\phi(z)$ consistent with the depicted assumptions will be, by definition, a distribution function in $[0,1]$. In our empirical specification we assume that $K(\bullet)$ belongs to the Beta distribution family, which offers a great deal of flexibility.

B. The Distribution of Bidders Valuations.

[13] Observe that if $\phi(1) = c$ we can always define $\Phi = c^{-1}\phi$ and $\delta_{ij} = c\lambda_{ij}$, with $\Phi(1) = 1$ and $\alpha_{ij} = \delta_{ij}\Phi\left(\bar{F}(b_{ij}(t))\right)$.

[14] Intuitively, bidders close to the lower end of the range of variation of v may be less likely to bid because they realize that their chances of winning the object are very small.

Bidders behavior is modeled in terms of the underlying distribution of bidders valuations and the following two behavioral assumptions: (1) Bidders do not bid more than they are willing to pay; (2) Close to the end of the auction those bidders with valuation of the item above its current minimum acceptable bid choose to bid their maximum willingness to pay, if they have not done that already. The first assumption is identical to that in Haile and Tamer (2003). The second assumption is somewhat stronger, it guarantees that the winning bid will be equal to the second highest willingness to pay in auctions with two or more active bidders. The second assumption in Haile and Tamer (2003) states that "Bidders do not allow an opponent to win at a price they are willing to beat." This assumption would be equivalent to ours under the traditional English auction framework employed in most empirical work, as long as bidders play according to their dominant strategy at some point before the end of the auction. After introducing some notation, we will argue that with the type of data available on eBay their assumption is only marginally different from ours.

Our assumptions are consistent with many different types of bidding behaviors. Bidders are free to bid as many times as they want and only at the end of the auction those remaining bidders with high enough valuation of the item are required to bid their true valuation, as postulated by assumption 2. In fact, in the econometric specification we will only require that the bidder with the second highest valuation bids her true valuation before the end of the auction and that the bidder with the highest valuation wins the auction.

Consider a particular auction that ends after time T with M bids, $\{b_k\}_{k=1}^{M}$, from K different bidders $(K \leq M)$. Assume also that associated to this auction are $N\,(\geq K)$ potential bidders with valuations $\{v_n\}_{n=1}^{N}$. Focusing our attention on the highest bid from each bidder, say $\{b_k^*\}_{k=1}^{K}$, and organizing this set in ascending order, we obtain

$$\bar{b}_1 < \bar{b}_2 < ... < \bar{b}_{K-1} < \max_k b_k^* = \bar{b}_K = \bar{b}_{K-1} + \Delta,$$

with $\bar{b}_k$ representing the k-th order statistic from the set $\{b_k^*\}_{k=1}^{K}$, $\bar{b}_{K-1} + \Delta$ representing the winning bid, and Δ representing the minimum increment allowed by eBay. Define also $\bar{v}_n$ as the n-th order statistic from the set $\{v_n\}_{n=1}^{N}$. Given assumption 1, Lemma 1 in Haile and Tamer (2003) guarantees that $\bar{b}_k \leq \bar{v}_{N-(K-k)}$, for $k = 1, ..., K$. In particular, for $k = K$ and $K - 1$, we have that $\bar{b}_K \leq \bar{v}_N$, and $\bar{b}_{K-1} \leq \bar{v}_{N-1}$. Furthermore, assumption 2 implies that $\bar{b}_{K-1} = \bar{v}_{N-1}$. Within this framework, the event $\left\{(b_k)_{k=1}^{M} | N\right\}$ has the same probability as the event $\left\{((\bar{b}_k \leq \bar{v}_{N-(K-k)})_{k=1}^{K}, \bar{b}_{K-1} = \bar{v}_{N-1} | N\right\}$, because assumptions 1 and 2 impose restrictions on the vector of maximum bids only.

Haile and Tamer (2003) construct their bounds based on two assumptions. Their first assumption is identical to ours. Their second assumption implies that if the winning bid is $\bar{b}_K$ the bidder with the second highest valuation would not be willing to beat this price, or $\bar{v}_{N-1} < \bar{b}_K$. This, along with assumption 1, implies that $\bar{b}_{K-1} < \bar{v}_{N-1} < \bar{b}_K = \bar{b}_{K-1} + \Delta$,or $\left|\bar{b}_{K-1} - \bar{v}_{N-1}\right| < \Delta$. Thus, taking into account that Δ takes a small value, it seems that not much is lost by implementing instead our assumption 2, which implies that $\bar{b}_{K-1} = \bar{v}_{N-1}$. In addition, our model can be manipulated more easily when conducting structural policy analysis.

In the empirical specification we need to take into account the presence of observed heterogeneity associated with each auction. Thus, denote by v_{ijk} bidder's k valuation of object j being auctioned by artist i and denote by X_{ij} the vector of characteristics specific to this object. We impose the following structure: $\log v_{ijk} = \phi\left(X_{ij}, \eta_{ij}\right) + w_{ijk}$, where w_{ijk} represents the residual, bidder specific, (log-)valuation of the object being auctioned. Finally, in our empirical specification we consider $\phi\left(X_{ij}, \eta_{ij}\right) = \beta_1 X_{ij} + \beta_2 \eta_{ij}$ and we assume that the distribution of unobserved valuations w_{ijk} is distributed normal. Any issues related to the unobservability of η_{ij} will be addressed at the end of the next subsection.

C. The Likelihood

We avoid mention of auction specific heterogeneity, except when absolutely necessary. Denote the time of the first bid by bidder k as t_k^1, for $k = 1, ..., K$. As described before, we assume that only the first bid conveys information about the arrival of new bidders. Also, the timing of any additional bid from an active bidder conveys no new information to active or potential bidders, other than its effect on the minimum acceptable bid. In this framework, the likelihood of a particular observation $(b_k, t_k)_{k=0}^{M}$, with b_0 representing the starting bid, can be computed as

$$P\left((b_k, t_k)_{k=0}^{M}\right) = P\left((t_k)_{k=1}^{M} \mid (b_k)_{k=0}^{M}\right) P\left((b_k)_{k=1}^{M}\right),$$

with $P\left((t_k)_{k=1}^{M} \mid (b_k)_{k=1}^{M}\right) = P\left((t_k^1)_{k=1}^{K} \mid (s_k\left(t_k^1\right))_{k=1}^{K}\right)$, and $s\left(t_k^1\right)$ representing the minimum acceptable bid an instant before a bid from a new bidder arrives at time t_k^1. On the other hand, the probability $P\left((b_k)_{k=1}^{M}\right)$ is a function of the potential number of bidders, which is unknown. However, we can determine the probability associated with a bid sequence of the form $\{b_k\}_{k=1}^{M}$ assuming that the number of potential bidders is N $(\geq K)$:

$$P\left((b_k)_{k=1}^{M}|N\right) = P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right) = P\left(\left(\left(\bar{b}_k \leq \bar{v}_{N-(K-k)}\right)_{k=1}^{K}, \bar{b}_{K-1} = \bar{v}_{N-1}|N\right)\right).$$

We can also determine the probability of N $(\geq K)$ potential bidders, $P(N|N \geq K, \lambda)$, conditional on the arrival process $P(\lambda)$. As a result, we can compute $P\left((b_k)_{k=1}^{M}\right)$ as

$$P\left(\left(\bar{b}_k\right)_{k=1}^{K}\right) = E_N\left(P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right)\right) = \sum_{N=K}^{\infty} P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right) P(N|N \geq K, \lambda).$$

The probability $P(N|N \geq K, \lambda)$ can be computed easily. Interestingly, the probability $P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right)$ also has an analytic solution, for all $K \geq 0$ and $N(\geq K)$ and for any $F(\bullet)$, which can be represented explicitly in the form of a recursive polynomial function of $\left\{F\left(\bar{b}_k\right)\right\}_{k=1}^{K-1}$, as is shown in appendix 2. With these results at hand, the computation of $P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right)$ is a numerical exercise that can be handled by any modern desktop computer.

Finally, the empirical specification described in previous subsections includes an unobserved heterogeneity component. We address this issue using the popular semi-parametric approach described in Heckman and Singer (1984). Subject to the standard assumptions of a random effects model, this approach hypothesizes a discrete distribution with finite support as a good approximation to the true distribution of the unobserved heterogeneity component. In this framework, the contribution to the likelihood of an observation $(b_k, t_k)_{k=0}^{M}$ can be computed as

$$P\left((b_k, t_k)_{k=0}^{M}\right) = \sum_{h=1}^{H} q_h P\left((b_k, t_k)_{k=0}^{M} |\eta_h\right)$$

with H representing the number of points of support of the discrete distribution, $q_h \geq 0$ the probability associated to a realization η_h, with $\sum_{h=1}^{H} q_h = 1$, and $\sum_{h=1}^{H} q_h\eta_h = 0$ a mean restriction necessary for identification. In the empirical work H is set to 2.[15]

4 Analysis of the Effects of 9/11.

In this section we apply our methodology to the analysis of the effects of the terrorist attacks of 9/11 in the market for art by self representing artists on eBay. Several authors have analyzed the effects of 9/11 in markets, industries and communities: financial markets (Straetmans, Verschoor and Wolff, 2003; Choudhry, 2003); the airline industry (Rupp, Holmes and DeSimone, 2003) and the automobile industry (Copeland and Hall, 2005), among others; and in communities like Israel or the Basque Country (Abadie and Gardeazabal, 2003; Eckstein and Tsiddon, 2004). Becker and Rubinstein (2004) take on a theoretical and empirical analysis of how terrorism influences peoples' behavior.

The strategy used in this paper to identify the effects of 9/11 is similar to that in Copeland and Hall (2005). We compare market outcomes before and after 9/11 with a similar period in 2004. This approach allows us to control for seasonality and overall changes in the market

[15] A simmilar assumption is made in many other papers using the Heckman-Singer approach. Baker and Melino (2000) in a Montecarlo study of a simple model show that this assumption usually works very well.

across years. The advantage of using a structural model to analyze the impact of 9/11 is that we can compute the effects of the shock on artists' and eBay's revenues as well as consumer welfare. In addition, the overall effect of the shock can be decomposed according to its effect on consumers' participation in the market (the arrival process) and consumers valuations. Furthermore, in this framework consumer welfare can be computed by simulating the difference between the buyers' valuation of an item and its transaction price.

Overall, our results are consistent with the findings of Copeland and Hall. They find a decrease in the production of cars in September 2001, but argue that this is mainly the result of "parts disruptions related to increased border security arising after September 11." Clearly, the artists in our data were not likely to be affected by this kind of restriction on inputs over the same period of time. Also, they find no evidence to support the claim that 9/11 had a negative effect on the demand for cars. In fact, they find that demand rose during the Autumn of 2001. Similarly, the final conclusion from our analysis is that the 9/11 shock had no significant impact on the supply of art and a short lived impact on the demand for art by self-representing artists on eBay. The results from this empirical exercise are discussed with detail in the next paragraphs.

A. The Supply of Art.

Several authors have modeled the labor supply of artists' theoretically (Throsby, 1994b; Cowen and Tabarrok, 2000; Caserta and Cuccia, 2001; see also Throsby, 1994a and Bryant and Throsby, 2005, for surveys) and empirically (Gapinski, 1980, 1984; Throsby, 2004). Keeping in mind that our data contains little information on artists' characteristics, we model artists' supply of artwork on eBay using a reduced form analysis. In particular, we consider several specifications of a negative binomial count model for two different dependent variables: the number of weekly paintings posted, and the weekly number of square feet posted for sale, by artists. The data consists of 840 artist/week observations from the last week of august to the first week of October, years 2001 and 2004. Figure 1 depicts the weekly average number of square feet supplied by the artists in our sample by year, and figure 2 depicts the weekly average selling price per square feet.

The artists in our data do not face the type of production-adjustment costs of car manufacturers. Thus, in contrast with Copeland and Hall's analysis, we can expect that any effect of the 9/11 shock on the production of art will became apparent the weeks after 9/11. Therefore, we define a dummy associated with the weeks after 9/11. Estimation results for the parameter associated to the 9/11 dummy are presented in table 6. All model specifications include weekly dummies to control for seasonal effects.[16] Models 1 and 3 also include as explanatory variables a measure of artist's reputation, and a year dummy, models 2 and 4 include the same explanatory variables plus artists specific fixed effects. As we anticipated, the coefficient associated to the 9/11 dummy is small in magnitude and insignificant in all models. Furthermore, the sign of the coefficient varies across models and is positive in three out of four models, which is not consistent with the assumption that 9/11 represented a negative shock on the supply of art on eBay.

B. The Demand for Art.

[16] Weekly dummies were defined such that 9/11 coincides with the last day of a particular week.

Table 7 presents estimation results for the empirical auction model described in the previous sections. Differences across auctions are captured by a vector of instruments. These instruments include: two measures of artists' reputation as defined in section 2, characteristics of the object being auctioned describing dimension, style, medium and ground, a dummy for year 2001 that captures any potential overall change in the market across years, weekly dummies intended to capture seasonal effects on demand that are common across years, and two dummy variables intended to capture the potential impact of the terrorists attacks of September 11 2001 on this market. The first 9/11 dummy is associated with auctions posted within one month after 9/11, while the second dummy is associated with any other auction posted in 2001 more than one month after 9/11. The empirical specification also includes artists' specific fixed effects to account for unobserved differences across artists, in addition to those captured by the vector of instruments, as well as an unobserved heterogeneity component as described in the previous section.

In addition, the process of arrival of new bidders includes also a baseline hazard that accounts for differences in the rate of arrival of new bidders at different times over the duration of the auction. This baseline hazard consists of a constant and four additional baseline parameters. The first baseline parameter accounts for differences in the arrival of bidders between the second day and the day before the end of the auction, the second, third and fourth baselines account for differences in the rate of arrival the last day, the last hour and the last ten minutes of the auction, respectively. Auctions of different length are accommodated by changing the length of the time interval between the second and last day of the auction. The process of arrival also includes dummies for the day of the week in which the auction ends, to allow for the possibility of changes in the flow of potential buyers for different days of the week. A likelihood ratio test cannot reject the assumption $K(z) = z$. Thus, the final estimation results reported in table 7 impose this restriction.

Looking at the arrival of bidders and object valuation equations we observe that the size of the painting has a significant positive effect on both equations.[17] Interestingly, the overall effect of the eBay feedback score, the percentage of positive feedback from buyers, is insignificant in both equations. In contrast, other authors find this variable to be significant (Melnik and Alm, 2002). It is possible to reconcile these apparently contradictory results. Observe that artists in our dataset have feedback rates above 98%, in fact a large number of artists have a perfect score. It is reasonable to think that buyers may find sellers in the range of 98 to 100 similarly trustworthy.[18] In contrast, the number of feedbacks from unique buyers has a significant positive, and non-linear, effect on the arrival of new bidders and a significant positive, and linear, effect on the valuation equation. It is possible that artists that have a large customer base also enjoy a larger number of repeated purchases from past buyers, or the flow of customers may increase due to a "word of mouth" effect.

Looking at the baseline hazard, we observe that bidders are more likely to bid the last day of the auction and less likely to bid between the first and last day. Furthermore, for the

[17] Table 5 reports larger paintings in average in 2004. This indicates a level of artists' response over time to customer tastes.

[18] The insignificant result associated to the eBay feedback score is not due to colinearity between the two different measures of reputation. We have estimated models including the eBay feedback variable as the only measure of reputation and have obtained the same insignificant results with a similar coefficient value, even in models in which artists' fixed effects are not included.

last day, bidders are more likely to participate close to the end of the auction. This result suggests the presence of inpatient bidders that only search among auctions that are about to end. The results also indicate a significantly higher rate of arrivals of new bidders in year 2001 as compared with year 2004. This is probably the result of a steady increase over time in the number of artists selling their work on eBay. There are other significant parameters in the model associated to style, medium and ground which we do not discuss.

The parameters most relevant to our study are those associated with the 9/11 dummies. These parameters are meant to capture unusual variations in demand one month and two to three months after 9/11, respectively. Looking first at the arrival equation, we observe that the coefficients associated to both dummies are positive, but insignificant. One possible interpretation of this result is that the events of 9/11 may have attracted the attention of "bargain hunters" looking for good deals. Looking now at the valuation equation we observe that the parameters associated to both dummies are negative, indicating a decrease in valuation, with only the first value significantly different from zero at the standard significance level. Simulations conducted using the model in table 7 suggest a small, but insignificant reduction in consumer surplus of about two percent, with little effect on other relevant measures, like artists' revenues.

5 Conclusions

This paper presents a novel structural approach for the estimation of IPV, ascending-bid, second-price auctions, the most popular type of auction mechanism on Internet markets.

In this approach both the arrival of bidders and the distribution of bidders valuations are modeled explicitly. We apply this procedure to the analysis of the market of art by self-representing artists on eBay. Our approach takes into account the specific features of eBay auctions: the number of potential bidders is stochastic, a function of the auction characteristics, and unobserved in many cases, and not all potential bids are always observed.

The model is estimated by maximum likelihood. Even though the contribution of each observation to the likelihood function entails solving a multivariate integral of a complex multivariate density function, we show that this can be done explicitly by making use of recursive polynomials, which simplifies the numerical analysis significantly. This result is general and does not rely on any specific functional form assumption about the distribution of bidders valuations. As a result, standard parametric and semiparametric methods can be implemented easily.

In an empirical exercise we apply this novel methodology to the analysis of the effects of the 9/11 shock in the market for art on eBay. The final conclusion from our analysis is that the 9/11 shock had no significant impact on the supply of art and a small and short lived impact on the demand for art by self-representing artists on eBay.

References

Abadie, Alberto and Javier Gardeazabal (2003). "The Economic Costs of Conflict: A Case Study of the Basque Country," *American Economic Review*, vol. 93(1), 113-132.

Athey, S. and P. A. Haile (2002). "Identification of Standard Auction Models," *Econometrica*, Vol. 70, No. 6, pp. 2107-2140.

Athey, S. and P. A. Haile (2005). "Nonparametric Approaches to Auctions," *Handbook of Econometrics VI*, forthcomming.

Ashenfelter, O. and K. Graddy (2002): "Art Auctions: A Survey of Empirical Studies," CEPR Discussion Paper 3387.

Ashenfelter, O. and K. Graddy (2003): "Auctions and the Price of Art," *Journal of Economic Literature*, September.

Bajari, P. and A. Hortacsu (2003): "The Winner's Curse, Reserve Prices, and Endogenous Entry: Empirical Insights from eBay Auctions," *The Rand Journal of Economics*, Vol. 34, No. 2, pp. 329-355.

Bajari, P. and A. Hortacsu (2004): "Economic Insights from Internet Auctions," *Journal of Economic Literature*, Vol. XLII, 457-486.

Baker, M. and A. Melino (2000) , "Duration Dependence and Nonparametric Heterogeneity: A Monte Carlo Study," *Journal of Econometrics* 96, 357-393.

Baldwin, L. H.; Marshall, R. C.; and J. F. Richard (1997): "Bidder Collusion and Forest Service Timber Sales," *Journal of Political Economy* 105(4), 657-99.

Becker, Gary and Yona Rubinstein (2004) , "Fear and the Response to Terrorism: an Economic Analysis." Working Paper, University of Chicago.

Bryant, W. D. A. and D Throsby (2005). "The economics of creativity in the arts", in Victor Ginsburgh and David Throsby (eds.), *Handbook of the Economics of Art and Culture*, (Amsterdam: North-Holland, forthcoming).

Cameron, A. C. and P. Trivedi (1998) . "Analysis of Count Data." *Econometric Society Monograph* No. 30, Cambridge University Press, 1998.

Canals-Cerda, J. (2005). "Congestion Pricing in an Internet Market." NET Institute Working Paper No. 05-10.

Caserta, M. and T. Cuccia (2001). "The Supply of Arts Labour: Towards a Dynamic Approach." *Journal of Cultural Economics* 25, 185-201, 2001.

Choudhry, T. (2002). "September 11 and Time-Varying Beta of United States Companies." Working Paper, School of Management, University of Bradford.

Copeland, A. and G. Hall (2005) "The Response of Prices, Sales, and Output to Temporary Changes in Demand," Working Paper, Yale University.

Cowen, T. and A. Tabarrok (2000). "An Economic Theory of Avant-Garde and Popular Art, or High and Low Culture," *Southern Economic Journal*, Vol. 67, No. 2, pp. 232-253.

Donald, S. G., and Paarsch, H. J. (1996). "Identification, Estimation and Testing in Parametric Empirical Models of Auctions within the Independent Private Values Paradigm." *Econometric Theory* 12, 517-67.

Z. Eckstein and D. Tsiddon 2004. "Macroeconomic Consequences of Terror: Theory and the Case of Israel." *Journal of Monetary Economics*, V 51:5, pp. 971-1002.

Elbers, C. and G. Ridder (1983). "True and Spurious Duration Dependence: the Identifiability of the Proportional Hazard Model." *Review of Economic Studies*, vol XLIX, pp. 403-409.

Flinn, C. and J. J. Heckman (1982). "New Methods for Analyzing Structural Models of Labor Force Dynamics." *Journal of Econometrics*. 18(1): 115-168.

Gapinski, J. H. (1980). "The Production of Culture." *The Review of Economics and Statistics*, Vol. 62, No. 4, 578-586.

Gapinski, J. H. (1984). "The economics of performing Shakespeare", *American Economic Review*, 74 (3): 458-466.

Haile, P. A. (2001): "Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales," *American Economic Review* 91(3), 399-427.

Haile, Philip A. and Elie T. Tamer (2003). "Inference with an Incomplete Model of English Auctions." *Journal of Political Economy*, vol. 111, no. 1: 1-51.

Hasker, K.; Gonzalez, R. and R. C. Sickles (2001): "An Analysis of Strategic Behavior and Consumer Surplus in eBay Auctions," Working Paper, Rice University.

J. Heckman and B. Singer (1984) "A Method for Minimizing the Impact of Distributional Assumptions in Econometric Models for Duration Data." *Econometrica*, Vol. 52, No. 2., pp. 271-320.

Hendricks, K. and H. J. Paarsch (1995): "A Survey of Recent Empirical Work Concerning Auctions." *Canadian Journal of Economics*, 28 (May), 403-26.

Hendricks, K. and R. H. Porter (2000) "Lectures on Auctions: An Empirical Perspective," mimeo, Northwestern University.

Hogg, R. V. and A. T. Craig (1978). "Introduction to Mathematical Statistics." Macmillan ed. Fourth Edition.

Hong, H. and M. Shum (2003): "Econometric Models of Asymmetric Ascending Auctions," *Journal of Econometrics*, 112: 327-358.

Klemperer, P. (1999). "Auction Theory: A Guide to the Literature." *Journal of Economics Surveys* 13, 227-286.

Laffont, Jean-Jacques, Herve Ossard, Quang Vuong (1995). "Econometrics of First-Price Auctions." *Econometrica*, Vol. 63, No. 4: 953-980.

Lucking-Reiley, D. (2000), "Auctions on the Internet: What's Being Auctioned, and How?" *Journal of Industrial Economics*, 48, 227-252.

Melnik, M. and J. Alm (2002): "Does a Seller's eCommerce Reputation Matter? Evidence from eBay Auctions," *Journal of Industrial Economics*, Vol. 50 (3), 337-349.

Milgrom, P. R. and R. J. Weber (1982). "A Theory of Auctions and Competitive Bidding." *Econometrica*, 50, 1089-1122.

Paarsch, H. J. (1992): "Empirical Models of Auctions and an Application to British Columbian Timber Sales," Research Report n. 9212. London: Univ. Western Ontario.

Paarsch, H. J. (1997): "Deriving an estimate of the optimal reserve price: An application to British Columbian timber sales," *Journal of Econometrics* 78, 333-357.

Perrigne I. and Q. Vuong (1999): "Structural Econometrics of First-Price Auctions: A Survey," *Canadian Journal of Agricultural Economics*, 47, 203-223.

Resnick, Paul and Richard Zeckhauser (2002), "Trust Among Strangers in Internet Transactions: Empirical Analysis of eBay's Reputation System," The Economics of the Internet and E-Commerce,Michael R. Baye, Editor, *Advances in Applied Microeconomics*, Vol. 11.

Roth, A. E. and A. Ockenfels (2002): "Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet," *American Economic Review*, 92 (4), 1093-1103.

Rupp, N. G., G. M. Holmes and Jeff DeSimone: "Airline Schedule Recovery After Airport Closures: Empirical Evidence Since September 11th." NBER Working Paper 9744.

Song, Unjy (2004). "Nonparametric Estimation of an eBay Auction Model with an Unknown Number of Bidders," Working Paper, University of Wisconsin at Madison.

Straetmans, S.T.M, W.F.C. Verschoor, and C.C.P. Wolff, (2003). "Extreme US Stock Market Fluctuations in the Wake of 9/11." Working Paper, Maastricht University.

Throsby, D. (1994a): "The Production and Consumption of the Arts: A View of Cultural Economics." *Journal of Economic Literature*, Vol. 32(1), pp. 1-29.

Throsby, D. (1994b): "A Work-Preference Model of Artist Behavior." in A. Peacock and I. Rizzo (eds.), *Cultural Economics and Cultural Policies*. Kluwer Academic Publishers, Dordrecht.

Throsby, D. (2004): “An artistic Production Function: Theory and an Application to Australian Visual Artists.” Working Paper, Macquarie University, Sydney, Australia.

Wang, R (1993) , “Auctions versus Posted-Price Selling,” *The American Economic Review*, Vol. 83, No. 4: 838-851.

Table 1: Frequency table for the number of bidders and bids across auctions.

	July to November 2001									
	Bidders									
Range	0	1	2	3	4	5	6	7	8	9-14
Frequency	852	470	370	238	140	68	48	26	12	25
% Frequency	37.88	20.9	16.45	10.58	6.22	3.02	2.13	1.16	0.53	1.1
	Bids									
Range	0	1	2	3	4	5	6	7	8-15	16-52
Frequency	852	456	186	125	125	101	89	49	216	50
% Frequency	37.88	20.28	8.27	5.56	5.56	4.49	3.96	2.18	9.61	2.2
	August to December 2004									
	Bidders									
Range	0	1	2	3	4	5	6	7	8	9-17
Frequency	940	409	223	177	150	115	84	74	41	56
% Frequency	41.43	18.03	9.83	7.8	6.61	5.07	3.7	3.26	1.81	2.46
	Bids									
Range	0	1	2	3	4	5	6	7	8-15	16-49
Frequency	940	369	142	92	80	65	64	51	262	204
% Frequency	41.43	16.26	6.26	4.05	3.53	2.86	2.82	2.25	11.56	8.97

Table 2: Frequency table for the duration of auctions.

	July to November 2001			
Range	3 days	5 days	7 days	10 days
Frequency	33	85	1586	545
% Frequency	1.47	3.78	70.52	24.23
	August to December 2004			
Range	3 days	5 days	7 days	10 days
Frequency	73	240	1116	840
% Frequency	3.22	10.58	49.18	37.02

Table 3: Distribution of paintings across artists

# Paintings	15-20	20-30	30-50	50-100	100+
# Artists	4	8	6	9	15

Table 4: Time remaining in the auction at the time of arrival of winning bid.

	July to November 2001							
Range	5 sec.	15 sec.	25 sec.	60 sec.	10 min.	1 Hour	1 Day	+1 Day
Frequency	16	45	33	70	106	114	418	595
% Freq.	1.15	3.22	2.36	5.01	7.59	8.16	29.92	42.59
C. Freq.	1.15	4.37	6.73	11.74	19.33	27.49	57.41	100.00
	August to December 2004							
Range	5 sec.	15 sec.	25 sec.	60 sec.	10 min.	1 Hour	1 Day	+1 Day
Frequency	44	115	45	71	139	114	371	430
% Freq.	3.31	8.65	3.39	5.34	10.46	8.58	27.92	32.36
C. Freq.	3.31	11.96	15.35	20.69	31.15	39.73	67.64	100.00

Table 5: Descriptive Statistics of Relevant Variables.

	July to November 2001			**August to December 2004**		
	Average	**Minimum**	**Maximum**	**Average**	**Minimum**	**Maximum**
Width	1.29	0.13	6.00	1.52	0.13	6.00
Height	1.29	0.08	5.50	1.45	0.17	6.00
Square Feet	2.06	0.01	20.25	2.71	0.02	24.00
eBay Feedback	99.9	98.36	100.00	99.78	98.98	100.00
# of unique feedbacks	1.34	0.00	5.60	5.09	0.39	16.50
Shipping cost	10.81	0.00	60.00	13.25	0.00	200.00
Selling Price	45.34	2.08	655.01	110.03	3.00	1449.00
N. Obs. (% Sold)		2249 (62 %)			2269 (57 %)	

Note: the '# of unique feedbacks' variable is measured in hundreds of feedbacks by unique users.

Table 6: Results from Count Models of the Artists' Weekly Supply of Art.

	Number of Weekly Paintings				**Weekly Paintings in Sq. feet**			
9/11 dummy	**Model 1**		**Model 2**		**Model 3**		**Model 4**	
	Parameter	T-val.	Parameter	T-val.	Parameter	T-val.	Parameter	T-val.
Parameter	.09434	0.55	.10971	0.92	.41561	1.38	-.00078	-0.00
Marginal Effect	.01013	0.50	.00003	0.00	.01788	0.35	-.00000	-0.00
LLF	-1239.50		-992.016		-1452.13		-1100.99	

Figure 1
Artists' Average Weekly Supply of Art in Square Feet

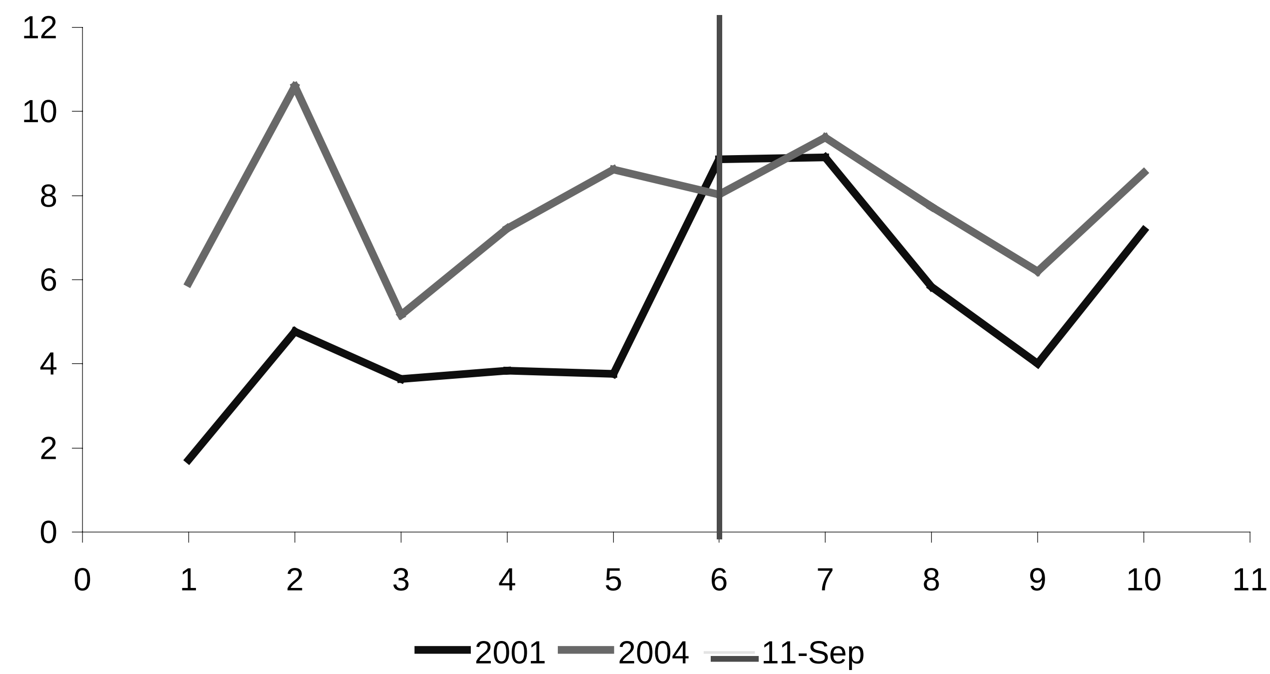

Figure 2: Weekly Average Price Per Sq. Feet

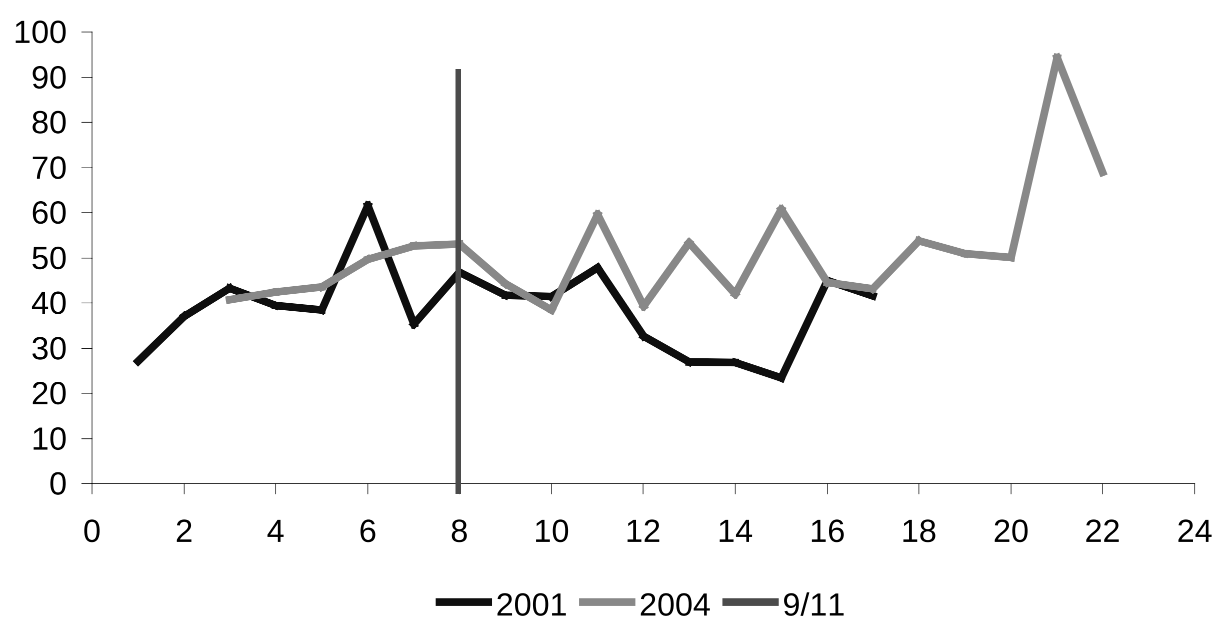

Table 7.- Estimation results for the model with artists' specific fixed effects.

	Bid Arrival		Valuation	
	Parameter	Std. error	Parameter	Std. error
Intercept	-15.0072	0.0000	1.8064	7.8027
Baseline1	-0.5543	0.0000		
Baseline2	0.6773	0.0009		
Baseline3	2.3943	0.0000		
Baseline4	5.0074	0.0470		
SEPTEMBER 11 DUMMY				
One Month After	0.0871	0.0973	-0.1056	0.0488
Two/Three Months After	0.0287	0.1056	-0.0476	0.0526
Year 2001 dummy	0.3610	0.1357	-0.1360	0.0703
FEEDBACK				
eBay Feedback	-0.0061	0.0061	0.0106	0.0781
# of unique feedbacks	0.0872	0.0380	0.0183	0.0206
# of unique feedbacks^2	-0.0061	0.0021	-0.0007	0.0011
DIMENSION				
Square-Feet	0.2216	0.0180	0.1349	0.0092
Square-Feet^2	-0.0116	0.0011	-0.0034	0.0006
STYLE				
Abstract	-0.2761	0.0683	-0.1046	0.0358
Contemporary	0.2404	0.0543	-0.1681	0.0290
Cubist	0.1373	0.0638	0.0526	0.0320
Folk	0.2001	0.0611	-0.0082	0.0345
Modern	0.0593	0.0579	0.0915	0.0312
Urban	0.0254	0.0659	-0.0801	0.0371
Pop	-0.1880	0.0597	0.1204	0.0309
MEDIUM				
Acrylic	0.4182	0.0842	-0.0437	0.0433
Pen and Ink	0.0825	0.1531	-0.1263	0.0971
Collage	0.2406	0.0896	-0.0902	0.0455
Oil	0.0219	0.0976	0.2449	0.0516
GROUND				
Stretch Canvas	0.2512	0.0870	0.2775	0.0417
Canvas Panel	0.3008	0.0900	0.0144	0.0456
Cardboard	0.1850	0.1177	-0.2178	0.0692
Paper	0.3176	0.0906	-0.3382	0.0451
OTHER				
Weekly Dummy	Yes		No	
Day of the Week Dummy	Yes		Yes	
Artists' Fixed Effects	Yes		Yes	
	N. Obs:	**4518**	**LLF:**	**-135495.05**

Note: the '# of unique feedbacks' variable is measured in hundreds of feedbacks by unique users.

6 Appendix 1.

This appendix presents identification results for different versions of the auction model described in section 3. The proofs make use of a well known result from the literature of count models (Cameron and Trivedy, 1998): the waiting time between two events of a Poisson process with associated parameter μ is described by a duration model with associated hazard μ.

Proposition 1 *Consider the homogeneous model in section 2 characterized by* $(\lambda(t), F(v))$, *and assume that the opening bid* $b_0 = 0$ *for a positive proportion of auctions.*[19] *Then the model is identified if we observe the transaction price (second highest bid) and the time of arrival of first bids by active bidders.*

Proof.-

We show first identification of $\lambda(t)$. Consider auctions with $b_0 = 0$ that have not received any bids by time t, then the observed instantaneous probability of arrival of a first bid at time t, conditional on no arrivals prior to t, is $\lambda(t)$. Consider now the observed distribution of second highest bids and the distribution of second highest bids conditional on n potential bidders, denoted $F^{(2)}$ and $F^{(n-1:n)}$, then

$$F^{(2)} = \sum_{n \geq 2} p_n F^{(n-1:n)}$$

[19] *This is not an unreasonable assumption. Many art auctions on ebay have starting price equal to one cent.*

with p_n representing the probability of n potential bidders,which is identified because $\lambda(t)$ is identified. Thus, because the above equation defines a unique relationship between $F^{(2)}$ and $\left\{F^{(n-1:n)} : n \geq 2\right\}$ and there exists a unique relationship between $F^{(n-1:n)}$ and $F(v)$, see Hogg and Craig (1978), we have that $F(v)$ is also identified. More precisely, consider $G(v)$ such that $G(v^*) < F(v^*)$, or $G(v^*) > F(v^*)$, for some v^* in the range of $F(\bullet)$, then $G^{(n-1:n)}(v^*) < F^{(n-1:n)}(v^*)$ for $n \geq 2$ and as a result $G^{(2)}(v^*) < F^{(2)}(v^*)$.[20]

Consider now an auction with price v at time t, the observed instantaneous probability of arrival of a first bid at time t, conditional on no arrivals prior to t, is $\lambda(t) K\left(\bar{F}(v)\right)$ and because all elements in this expression are identified except for $K(\bullet)$, using standard arguments, variation in $F(v)$ identifies $K(\bullet)$.

Q.E.D.

Proposition 2 *Consider the heterogeneous model in section 2 characterized by the pair* $(\lambda(t,X), K(z), F(v|X))$*, such that* $v_i = g(X) w_i$ *with* w_i *iid, and assume that the opening bid* $b_0 = 0$ *for a positive proportion of auctions. Then the model is identified if we observe the transaction price (second highest bid) and the time of arrival of first bids by active bidders.*

Proof.-

As with the previous proposition, identification of $\lambda(t,X)$ is obtained from standard identification results in single spell hazard models. Also, theorem 3.i in Athey and Haile (2002) establishes that for n known, $\{v_i\}_{i=1}^{n}$ independent conditional on X, and the transaction

[20] Observe that the probability of a random draw falling in the range $[0, v^*]$ is smaller if $G(.)$ is the associated distribution. Thus, the probability of the $(n, n-1)$ statistic falling in range $[0, v^*]$ shold also be smaller in that case.

price is observed then $F(v|X)$ is identified.

In our case, n is not known. However, we can establish the following relationship between the observed distribution of second highest bids, denoted$F^{(2)}(v|X)$, and the distribution of second highest bids conditional on n potential bidders, $F^{(n-1:n)}(v|X)$,

$$F^{(2)}(v|X) = \sum_{n\geq 2} p_{n,X} F^{(n-1:n)}(v|X)$$

with $p_{n,X}$ representing the probability of n potential bidders, which is identified. Thus, taking into account that the above equation defines a unique relationship between $F^{(2)}(v|X)$ and $\left\{F^{(n-1:n)}(v|X) : n \in N\right\}$, and there exists a unique relationship between $F^{(n-1:n)}(v|X)$ and $F(v|X)$, we have that $F(v|X)$ is also identified.

Identification of $K(\bullet)$ can be shown using the same arguments from proposition 1.

Q.E.D.

In the next proposition we consider identification of a model that allows for unobserved heterogeneity in the arrival of bids that does non directly affect valuations.

Proposition 3 *Consider the heterogeneous model in section 2 characterized by the vector* $((\lambda(t|X,u), K(z), G(u)), F(v|X))$, *with* $\lambda(t|X,u) = \lambda(t,X)u$. *This model is identified under the same assumptions of the previous proposition.*

Proof.-

The only difference with respect to the previous proposition is the existence of unobserved heterogeneity in the arrival of bids. Focusing our attention on the arrival of bids as in proposition 1, we can apply standard identification results for proportional hazard models with unobserved heterogeneity (see for example Elbers and Ridder, 1983), that guarantee non parametric identification of $\lambda(t, X)$ and $G(u)$ up to location. This guarantees identification of $\{p_{n,X}\}_n$. Identification of $F(v|X)$ can be achieved using the same arguments as in the previous proposition.

Consider now an auction with price v at time t, the observed instantaneous probability of arrival of a first bid at time t, conditional on not arrivals prior to t, is $\lambda(t) K\left(\bar{F}(v|X)\right)$ and because all elements in this expression are identified except for $K(\bullet)$, using standard arguments, variation in $F(v)$ identifies $K(\bullet)$.

Q.E.D.

In the next proposition we consider identification of a model that allows for the presence of unobserved heterogeneity in the arrival of bids and bidders' valuations.

Proposition 4 *Consider the heterogeneous model in section 2 characterized by the vector* $((\lambda(t|X, u), K(z), G(u)), F(v|X, u, \delta))$, *with* $\lambda(t|X, u) = \lambda(t, X) u$ *and* $lnv_X = \phi(X) + \delta u + w$. *This model is identified under the same assumptions of the previous proposition and the additional exclusion restriction assumption that* w *has a bounded support and there exists a continuous explanatory variable affecting the arrival of bids but not valuations.*

Proof.-

Consider auctions with $b_0 = 0$ and that have not received any bids by time t, then the observed instantaneous probability of arrival of a first bid at time t, conditional on not arrivals prior to t, is $\lambda(t|X, u)$. Thus, existing results for non-parametric identification of proportional hazard models (Elbers and Ridder, 1983) guarantee non-parametric identification of $(\lambda(t|X), G(u))$, up to a proportionality normalization.

Assume now that the transaction price and the number of bidders are observed. That is, we observe $F_v^{(n-1,n)}(z|X)$, which is equal to $F_w^{(n-1,n)}(\ln z - \phi(X) - \delta u|X)$ if u is observed, or equal to

$$E_u\left[F_w^{(n-1,n)}(\ln z - \phi(X) - \delta u|X)|\lambda\right]$$

in general. In the above expression the conditional expectation is required because, due to selection, the distribution of u associated to auctions with n bidders depends on the value of $\lambda(t|X)$. Next we show identification of $\phi(\bullet)$ without specific knowledge of $(F_w(\bullet), G(\bullet))$. Consider X and X_1 with the same associated baseline hazard λ, this is made possible by the exclusion restriction assumption. Consider also z, z_1 such that $F_v^{(n-1,n)}(z|X) = F_v^{(n-1,n)}(z_1|X_1)$, then because $F_v^{(n-1,n)}(\bullet|X_1)$ is strictly increasing, it should be the case that $z - \phi(X) = z_1 - \phi(X_1)$, or $z_1 - z = \phi(X_1) - \phi(X)$. Using the last expression, and standard arguments, variation in z and X identify $\phi(\bullet)$ up to a location normalization.

In general we do not observe n, but we observe $F_v^{(2)}(z|X)$ which is strictly increasing as a function of $(z - \phi(X))$ and equal to

$$\sum_{n\geq 2} p_{n,\lambda} E_u\left[F_w^{(n-1,n)}(\ln z - \phi(X) - \delta u|X)|\lambda, n\right],$$

with $\{p_{n,\lambda}\}$ identified. We can apply the same arguments as before to this expression in order to obtain identification of $\phi(\bullet)$ up to a location normalization.

Consider now identification of δ. Observe that, $\ln v_X^{(n-1,n)} = \phi(X) + \delta u_X + w^{(n-1,n)}$,

$$
\begin{aligned}
E\left(\ln v^{(n+k-1,n+k)}|X_1\right) - \phi(X_1) &= \delta E(u|X_1, n+k) + E\left(w^{(n+k-1,n+k)}\right), \\
E\left(\ln v^{(n-1,n)}|X_2\right) - \phi(X_2) &= \delta E(u|X_2, n) + E\left(w^{(n-1,n)}\right),
\end{aligned}
$$

and $E(u|X,n)$ is identified for any pair (X,n). After substracting the second equation from the first one, the left hand side of the resulting equation is identified and the right hand side is equal to

$$
\delta\left[E(u|X_1, n+k) - E(u|X_2, n)\right] + \left[E\left(w^{(n+k-1,n+k)}\right) - E\left(w^{(n-1,n)}\right)\right] \tag{3}
$$

with $[E(u|X_1, n+k) - E(u|X_2, n)]$ identified and different from zero in most cases, unless u has a degenerated distribution, and $\left[E\left(w^{(n+k-1,n+k)}\right) - E\left(w^{(n-1,n)}\right)\right]$ converging to zero as n increases. Thus, for a large enough n this second term can be made negligible when compared with $[E(u|X_1, n+k) - E(u|X_2, n)]$. This proves identification of δ when n is known. In general the number of potential bidders n is not known but the number of actual bidders $m\,(\leq n)$ is known. Thus for m large enough we have that $\left[E\left(w^{(n+k-1,n+k)}\right) - E\left(w^{(n-1,n)}\right)\right] < \varepsilon/2$

$\forall n \geq m$, and, for $j = m, m+1$, $E\left(w^{(n-1,n)}|j\right)$ is equal to

$$\sum_{n \geq j} p\left(n|j,\lambda\right) E\left(w^{(n-1,n)}\right) \in \left(E\left(w^{(m-1,m)}\right) - \varepsilon/2, E\left(w^{(m-1,m)}\right) + \varepsilon/2\right),$$

as a result, $\left[E\left(w^{(n-1,n)}|m+1\right) - E\left(w^{(n-1,n)}|m\right)\right] < \varepsilon$ and the argument stated above for the case of a known number of potential bidders can also be applied when only the number of active bidders is observed. This proves identification of δ.

Next we show identification of $F_v\left(\bullet|X\right)$. In this case, we can establish the following relationship between the observed distribution of second highest bid and the distribution of second highest bid conditional on n potential bidders, denoted $F^{(2)}\left(v|X\right)$ and $F^{(n-1:n)}\left(v|X\right)$,

$$F^{(2)}\left(v|X\right) = \sum_{n \geq 2} p_{n,X} \int F^{(n-1:n)}\left(v|X,u\right) dG\left(u\right).$$

Taking into account that the above equation defines a unique relationship between $F^{(2)}\left(v|X\right)$ and $\left\{F^{(n-1:n)}\left(v|X\right) : n \geq 2\right\}$, using the same argument as in the previous propositions, identification of $F_v\left(\bullet|X\right)$ follows. It remains to show identification of $K\left(\bullet\right)$.

Consider $S_K\left(t|X,v\right) = E_u\left(\exp\left(-K\left(\bar{F}\left(v|X,u\right)\right)\Lambda\left(t|X\right)u\right)\right)$ representing the survival function for an auction with associated values (X,v) and integrated baseline hazard $\Lambda\left(t|X\right)$ at time t. If $K\left(\bullet\right)$ is not identified then there exists $H\left(\bullet\right) = K\left(s\right) + \Delta\left(s\right)$, with $\Delta\left(s\right) \neq 0$ in a set of non-zero measure, such that $S_K\left(t|X,v\right) = S_H\left(t|X,v\right)$ for any proper (t,X,v). In this framework, it must be the case that, for some value u^* in the support of u and for some value v^* in the support of $\tilde{v}_X, S_K(t|X, v^*, u \leq u^*) \neq S_H(t|X, v^*, u \leq u^*)$, and

$$\begin{aligned} S_K(t|X, v^*, u \leq u^*) + S_K(t|X, v^*, u > u^*) &= S_H(t|X, v^*, u \leq u^*) + S_H(t|X, v^*, u > u^*), \\ \text{or } S_K(t|X, v^*, u \leq u^*) - S_H(t|X, v^*, u \leq u^*) &= S_H(t|X, v^*, u > u^*) - S_K(t|X, v^*, u > u^*) \end{aligned}$$

for any $t \in [0, T]$. For a fixed X, consider $\bar{F}^*(u) = \bar{F}(v^*|X, u)$, $E_u^*(\bullet) = E_u(\bullet|X, u \leq u^*)$, $\bar{E}_u^*(\bullet) = E_u(\bullet|X, u > u^*)$ and $\Lambda^* = \Lambda(\bullet|X)$. Expanding the last equation we obtain:

$$\begin{aligned} & E_u^*\left(\exp\left[-K\left(\bar{F}^*(u)\right)u\right]^{\Lambda^*}\right) - E_u^*\left(\exp\left[-\left(K\left(\bar{F}^*(u)\right) + \Delta\left(\bar{F}^*(u)\right)\right)u\right]^{\Lambda^*}\right) \\ = \; & \bar{E}_u^*\left(\exp\left[-\left(K\left(\bar{F}^*(u)\right) + \Delta\left(\bar{F}^*(u)\right)\right)u\right]^{\Lambda^*}\right) - \bar{E}_u^*\left(\exp\left[-K\left(\bar{F}^*(u)\right)u\right]^{\Lambda^*}\right) \end{aligned}$$

and this equality must be satisfied for all $\Lambda^* \in [0, \Lambda(T|X)]$. We conclude by arguing, without proof, that this type of non-linear restriction can only be satisfied by a function $\Delta(\bullet) = 0$ and as a result $K(\bullet)$ is identified.

Q.E.D..

7 Appendix 2:

In order to compute $P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right)$ it is useful to first represent this probability in terms of the underlying density of bidder's valuations $f(v)$. From the relationship between order statistics and the originating random variable (Hogg and Craig, 1978, page 155) we have that $g(\bar{v}_1,...,\bar{v}_N) = N!f(\bar{v}_1)...f(\bar{v}_N)$, $\bar{v}_1 < \bar{v}_2 < ... < \bar{v}_N$, with $\{\bar{v}_n\}_{n=1}^{N}$ representing the set of order statistics from a random sample of N i.i.d realizations from a random variable with associated density $f(\cdot)$. When the number of potential bidders N coincides with the number of actual bidders K the probability $P\left(\left(\bar{b}_k\right)_{k=1}^{K}|K\right)$ is equal to,

$$K!f\left(\bar{b}_{K-1}\right)\int_{\bar{b}_K}^{+\infty} f\left(z_K\right)\int_{\bar{b}_{K-2}}^{\bar{b}_{K-1}} f\left(z_{K-2}\right)\int_{\bar{b}_{K-3}}^{z_{K-2}} f\left(z_{K-3}\right)...\int_{\bar{b}_1}^{z_2} f\left(z_1\right)dz_1...dz_K,$$

or

$$K\left(K-1\right)f\left(\bar{b}_{K-1}\right)\left(1-F\left(\bar{b}_K\right)\right)\left(K-2\right)!\int_{\bar{b}_{K-2}}^{\bar{b}_{K-1}} f\left(z_{K-2}\right)\int_{\bar{b}_{K-3}}^{z_{K-2}} f\left(z_{K-3}\right)...\int_{\bar{b}_1}^{z_2} f\left(z_1\right)dz_1...dz_{K-2}.$$

In general, the probability $P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right)$ will be equal to,

$$N!f\left(\bar{b}_{K-1}\right)\int_{\bar{b}_K}^{+\infty} f\left(z_N\right)\int_{\bar{b}_{K-2}}^{\bar{b}_{K-1}} f\left(z_{N-2}\right)\int_{\bar{b}_{K-3}}^{z_{N-2}} f\left(z_{N-3}\right)...\int_{\bar{b}_1}^{z_{N-K+2}} f\left(z_{N-K+1}\right)I\left(z_{N-K+1}\right)dz_{N-K+1}...dz_N$$

with $I\left(z_{N-K+1}\right)$ equal to

$$\int_{-\infty}^{z_{N-K+1}} f\left(z_{N-K}\right)\int_{-\infty}^{z_{N-K}} f\left(z_{N-K-1}\right)...\int_{-\infty}^{z_2} f\left(z_1\right)dz_1...dz_{N-K-1}dz_{N-K}.$$

The probability $P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right)$ can be decomposed into two components. The first one is

$$N\left(N-1\right)f\left(\bar{b}_{K-1}\right)\int_{\bar{b}_K}^{+\infty}f\left(z_N\right)dz_N=N\left(N-1\right)f\left(\bar{b}_{K-1}\right)\left(1-F\left(\bar{b}_K\right)\right).$$

The second one

$$(N-2)!\int_{\bar{b}_{K-2}}^{\bar{b}_{K-1}}f\left(z_{N-2}\right)\int_{\bar{b}_{K-3}}^{z_{N-2}}f\left(z_{N-3}\right)...\int_{\bar{b}_1}^{z_{N-K+2}}f\left(z_{N-K+1}\right)I\left(z_{N-K+1}\right)dz_{N-K+1}...dz_{N-2} \qquad (4)$$

interestingly also has an analytic solution which can be represented explicitly in the form of a polynomial function of $\left\{F\left(\bar{b}_k\right)\right\}_{k=1}^{K-1}$, as is shown in appendix 2.1. With these results at hand, the computation of $P\left(\left(\bar{b}_k\right)_{k=1}^{K}|N\right)$ is a straightforward numerical exercise that can be handled by any modern desktop computer.

Two special cases in which the above analysis does not apply is $K=0,1$. $K=0$ indicates that none of the potential bidders that had a chance to bid valued the object being auctioned above its starting bid. This probability can be computed as

$$P\left(b_0,0\right)=P\left(0|b_0\right)P\left(b_0\right),$$

with $P\left(0|b_0\right)$ representing the probability of arrival of zero bids over the duration of the auction, conditional on a minimum starting value of b_0, and $P\left(b_0\right)$ represents the probability of zero bidders with valuation higher or equal to b_0. In particular,

$$P(b_0) = \sum_{N=0}^{\infty} P(b_0|N)\, P(N|N \geq 0, \lambda)$$

with $P(b_0|N)$ representing the probability of N potential bidders with valuations below b_0, which is equal to $P(b_0|N) = P(\bar{v}_N \leq b_0) = F^N(b_0)$. On the other hand, the case $K = 1$ indicates that one active bidder was willing to bid above the minimum starting bid. In this case assumption 2 does not apply either,

$$\begin{aligned} P\left(\bar{b}_1 = b_0, t_1\right) &= P(t_1|b_0)\, P\left(\bar{b}_1 = b_0\right), \\ \textit{with}\ P\left(\bar{b}_1 = b_0\right) &= P(b_0 \leq \bar{v}_1|N = 1)\, P(N = 1|N \geq 1, \lambda) \\ &\quad + \sum_{N=2}^{\infty} P(\bar{v}_{N-1} \leq b_0 + \Delta, b_0 \leq \bar{v}_N|N)\, P(N|N \geq 1, \lambda) \end{aligned}$$

with $P(b_0 \leq \bar{v}_1|N = 1) = 1 - F(b_0)$ indicating that in the case of a unique potential bidder, and a unique active bidder, the bidder's valuation should be above b_0, and with $P(\bar{v}_{N-1} \leq b_0 + \Delta, b_0 \leq \bar{v}_N|N)$ indicating that in the case of N potential bidders, and a unique active bidder, the highest valuation should be above b_0 and all others should be below $b_0 + \Delta$. In this case also, the probability $P(\bar{v}_{N-1} \leq b_0 + \Delta, b_0 \leq \bar{v}_N|N)$ has an explicit solution as shown in appendix 2.2.

A.- Appendix 2.1

Next we show how to define an explicit recursive method for computing analytically the following integral

$$\int_{\bar{b}_{K-2}}^{\bar{b}_{K-1}} f\left(z_{N-2}\right) \int_{\bar{b}_{K-3}}^{z_{N-2}} f\left(z_{N-3}\right) ... \int_{\bar{b}_{1}}^{z_{N-K+2}} f\left(z_{N-K+1}\right) I\left(z_{N-K+1}\right) dz_{N-K+1}...dz_{N-2}, \tag{5}$$

denoted as $I\left[\left(\bar{b}_k\right)_{k=1}^{K-1} |N-2\right]$.

Consider first $I\left(z_{N-K+1}\right)$, equal to

$$\int_{-\infty}^{z_{N-K+1}} f\left(z_{N-K}\right) \int_{-\infty}^{z_{N-K}} f\left(z_{N-K-1}\right) ... \int_{-\infty}^{z_2} f\left(z_1\right) dz_1...dz_{N-K-1}dz_{N-K}.$$

Observe that the probability $P\left(\bar{v}_{N-K} \leq z_{N-K+1}|N-K\right)$ is equal to $(N-K)!I\left(z_{N-K+1}\right)$ and $P\left(\bar{v}_{N-K} \leq z_{N-K+1}|N-K\right)$ is equal to $F\left(z_{N-K+1}\right)^{N-K}$. Thus,

$$I\left(z_{N-K+1}\right) = (N-K)!^{-1} F\left(z_{N-K+1}\right)^{N-K}.$$

Consider now the following change of variables

$$F\left(z_j\right) = F_j \Rightarrow f\left(z_j\right) dz_j = dF_j,\ j = 1, 2 \text{ and denote } B_j = F\left(\bar{b}_j\right).$$

We have that $I\left(z_{N-K+1}\right) = (N-K)!^{-1} F_{N-K+1}^{N-K}$.

We consider now the analytic solution of $I\left[\left(\bar{b}_k\right)_{k=1}^{K-1} |N-2\right]$:

A.1.- We begin with an illustrative example $(K-1=5, N-2=6, N-K+1=3)$:

$$\int_{\bar{b}_4}^{\bar{b}_5} f\left(z_6\right) \int_{\bar{b}_3}^{z_6} f\left(z_5\right) \int_{\bar{b}_2}^{z_5} f\left(z_4\right) \int_{\bar{b}_1}^{z_4} f\left(z_3\right) I\left(z_3\right) dz_3 dz_4 dz_5 dz_6$$

$$= \int_{B_4}^{B_5} \int_{B_3}^{F_6} \int_{B_3}^{F_5} \int_{B_2}^{F_4} 2!^{-1} F_3^2 dF_3 dF_4 dF_5 dF_6$$

$$= \int_{B_4}^{B_5} \int_{B_3}^{F_6} \int_{B_2}^{F_5} \left[3!^{-1} x^3\right]_{B_1}^{F_4} dF_4 dF_5 dF_6 = \int_{B_4}^{B_5} \int_{B_3}^{F_6} \int_{B_2}^{F_5} 3!^{-1} \left[F_4^3 - B_1^3\right] dF_4 dF_5 dF_6$$

$$= \int_{B_4}^{B_5} \int_{B_3}^{F_6} \left[p_{4,3}\left(F_5\right) - p_{4,3}\left(B_2\right)\right] dF_5 dF_6$$

$$\{\text{with } p_{4,3}\left(x\right) = 4!^{-1} x^4 - 3!^{-1} B_1^3 x\}$$

$$= \int_{B_4}^{B_5} \left[p_{5,3}\left(F_6\right) - p_{5,3}\left(B_3\right)\right] dF_6$$

$$\{\text{with } p_{5,3}\left(x\right) = 5!^{-1} x^5 - 3!^{-1} 2^{-1} B_1^3 x^2 - p_{4,3}\left(B_2\right) x\}$$

$$= \left[p_{6,3}\left(x\right)\right]_{B_4}^{F_5} = p_{6,3}\left(B_5\right) - p_{6,3}\left(B_4\right)$$

$$\{\text{with } p_{6,3}\left(x\right) = 6!^{-1} x^6 - 3!^{-1} 3!^{-1} B_1^3 x^3 - 2^{-1} p_{4,3}\left(B_2\right) x^2 - p_{5,3}\left(B_3\right) x\}$$

A.2.- In general,

$$\int_{\bar{b}_{K-2}}^{\bar{b}_{K-1}} f\left(z_{N-2}\right) \int_{\bar{b}_{K-3}}^{z_{N-2}} f\left(z_{N-3}\right)\ldots \int_{\bar{b}_1}^{z_{N-K+2}} f\left(z_{N-K+1}\right) I\left(z_{N-K+1}\right) dz_{N-K+1}\ldots dz_{N-2}$$

$$= \int_{B_{K-2}}^{B_{K-1}} \int_{B_{K-3}\ldots}^{F_{N-2}} \int_{B_1}^{F_{N-K+2}} (N-K)!^{-1} F_{N-K+1}^{N-K} dF_{N-K+1}\ldots dF_{N-2}$$

$$= \int_{B_{K-2}}^{B_{K-1}} \int_{B_{K-3}\ldots}^{F_{N-2}} \int_{B_2}^{F_{N-K+3}} (N-K+1)!^{-1} \left[x^{N-K+1}\right]_{B_1}^{F_{N-K+2}} dF_{N-K+2}\ldots dF_{N-2}$$

$$= \int_{B_{K-2}}^{B_{K-1}} \int_{B_{K-3}\ldots}^{F_{N-2}} \int_{B_3}^{F_{N-K+4}} \left[p_{N-K+2,N-K+1}\left(x\right)\right]_{B_2}^{F_{N-K+3}} dF_{N-K+3}\ldots dF_{N-2}$$

$$\{\text{with } p_{N-K+2,N-K+1}\left(x\right) = (N-K+2)!^{-1} x^{N-K+2} - (N-K+1)!^{-1} B_1^{N-K+1} x\}$$

$$= \int_{B_{K-2}}^{B_{K-1}} \int_{B_{K-3}\ldots}^{F_{N-2}} \int_{B_4}^{F_{N-K+5}} \left[p_{N-K+3,N-K+1}\left(x\right)\right]_{B_3}^{F_{N-K+4}} dF_{N-K+4}\ldots dF_{N-2}$$

$$\{\text{with } p_{N-K+3,N-K+1}\left(x\right) = (N-K+3)!^{-1} x^{N-K+3} - (N-K+1)!^{-1} 2^{-1} B_1^{N-K+1} x^2$$

$$-p_{N-K+2,N-K+1}\left(B_2\right) x\}$$

$$= \int_{B_{K-2}}^{B_{K-1}} \int_{B_{K-3}\ldots}^{F_{N-2}} \int_{B_5}^{F_{N-K+6}} \left[p_{N-K+4,N-K+1}\left(x\right)\right]_{B_4}^{F_{N-K+5}} dF_{N-K+5}\ldots dF_{N-2}$$

$$\{\text{with } p_{N-K+4,N-K+1}\left(x\right) = (N-K+4)!^{-1} x^{N-K+4} - (N-K+1)!^{-1} 3!^{-1} B_1^{N-K+1} x^3$$

$$-2^{-1} p_{N-K+2,N-K+1}\left(B_2\right) x^2 - p_{N-K+3,N-K+1}\left(B_3\right) x\}$$

$$= ... = \int_{B_{K-2}}^{B_{K-1}} \left[p_{N-3,N-K+1}(x)\right]_{B_{K-3}}^{F_{N-2}} dF_{N-2}$$

$$\{\text{with } p_{N-3,N-K+1}(x) = (N-3)!^{-1}x^{N-3} - (N-K+1)!^{-1}(K-4)!^{-1}B_1^{N-K+1}x^{K-4}$$

$$-(K-5)!^{-1}p_{N-K+2,N-K+1}(B_2)\,x^{K-5} - (K-6)!^{-1}p_{N-K+3,N-K+1}(B_3)\,x^{K-6}$$

$$... - 2^{-1}p_{N-5,N-K+1}(B_{K-5})\,x^2 - p_{N-4,N-K+1}(B_{K-4})\,x\}$$

$$= \left[p_{N-2,N-K+1}(x)\right]_{B_{K-2}}^{B_{K-1}}$$

$$\{\text{with } p_{N-2,N-K+1}(x) = (N-2)!^{-1}x^{N-2} - (N-K+1)!^{-1}(K-3)!^{-1}B_1^{N-K+1}x^{K-3}$$

$$-(K-4)!^{-1}p_{N-K+2,N-K+1}(B_2)\,x^{K-4} - (K-5)!^{-1}p_{N-K+3,N-K+1}(B_3)\,x^{K-5}$$

$$... - 3!^{-1}p_{N-5,N-K+1}(B_{K-5})\,x^3 - 2^{-1}p_{N-4,N-K+1}(B_{K-4})\,x^2 - p_{N-3,N-K+1}(B_{K-3})\,x\}$$

Thus, $I\left[\left(\bar{b}_k\right)_{k=1}^{K-1}|N-2\right]$ can be computed analytically as a polynomial function of $\left\{F\left(\bar{b}_k\right)\right\}_{k=1}^{K-1}$.

Finally,

$$(N-2)!I\left[\left(\bar{b}_k\right)_{k=1}^{K-1}|N-2\right] = \left[q_{N-2,N-K+1}(x)\right]_{B_{K-2}}^{B_{K-1}}$$

$$\text{with } q_{N-2,N-K+1}(x) = x^{N-2} - \begin{pmatrix} N-2 \\ K-3 \end{pmatrix} B_1^{N-K+1} x^{K-3}$$

$$- \begin{pmatrix} N-2 \\ K-4 \end{pmatrix} q_{N-K+2,N-K+1}(B_2)\, x^{K-4} - \begin{pmatrix} N-2 \\ K-5 \end{pmatrix} q_{N-K+3,N-K+1}(B_3)\, x^{K-5}$$

$$\ldots - \begin{pmatrix} N-2 \\ 3 \end{pmatrix} q_{N-5,N-K+1}(B_{K-5})\, x^3 - \begin{pmatrix} N-2 \\ 2 \end{pmatrix} q_{N-4,N-K+1}(B_{K-4})\, x^2$$

$$- \begin{pmatrix} N-2 \\ 1 \end{pmatrix} q_{N-3,N-K+1}(B_{K-3})\, x\}$$

$$\text{and } q_{N-j,N-K+1}(x) = (N-j)!\, p_{N-j,N-K+1}(x)$$

B.- Appendix 2.2:

Next we show how to compute $P(\bar{v}_{N-1} \leq b_0 + \Delta, b_0 \leq \bar{v}_N | N)$. We could do that by following the same approach as in part one of this appendix. A more direct approach makes use of the expression of the joint distribution of any two order statistics (Hogg and Craig, 1978, page 160), in particular for $(N, N-1)$:

$$g_{N,N-1}(v_N, v_{N-1}) = N(N-1)\, F(v_{N-1})^{N-2} f(v_{N-1})\, f(v_N), \; v_{N-1} \leq v_N$$

and $P\left(\bar{v}_{N-1} \leq b_0 + \Delta, b_0 \leq \bar{v}_N | N\right)$

$$
\begin{gathered}
= \int_{b_0}^{+\infty} \int_{-\infty}^{b_0+\Delta} g_{N,N-1}\left(z_N, z_{N-1}\right) dz_{N-1} dz_N \\
= \int_{b_0}^{+\infty} \left\{ \int_{b_0}^{\min\{b_0+\Delta, z_N\}} g_{N,N-1}\left(z_N, z_{N-1}\right) dz_{N-1} + \int_{-\infty}^{b_0} g_{N,N-1}\left(z_N, z_{N-1}\right) dz_{N-1} \right\} dz_N \\
= \int_{b_0}^{+\infty} \int_{-\infty}^{b_0} g_{N,N-1}\left(z_N, z_{N-1}\right) dz_{N-1} dz_N + \\
\int_{b_0}^{b_0+\Delta} \int_{b_0}^{z_N} g_{N,N-1}\left(z_N, z_{N-1}\right) dz_{N-1} dz_N + \int_{b_0+\Delta}^{+\infty} \int_{b_0}^{b_0+\Delta} g_{N,N-1}\left(z_N, z_{N-1}\right) dz_{N-1} dz_N \\
= N(N-1) \{ \int_{b_0}^{+\infty} \int_{-\infty}^{b_0} F\left(z_{N-1}\right)^{N-2} f\left(z_{N-1}\right) dz_{N-1} f\left(z_N\right) dz_N + \\
\int_{b_0}^{b_0+\Delta} \int_{b_0}^{z_N} F\left(z_{N-1}\right)^{N-2} f\left(z_{N-1}\right) dz_{N-1} f\left(z_N\right) dz_N + \\
\int_{b_0+\Delta}^{+\infty} \int_{b_0}^{b_0+\Delta} F\left(z_{N-1}\right)^{N-2} f\left(z_{N-1}\right) dz_{N-1} f\left(z_N\right) dz_N \} \\
= N(N-1) \{ I_1 + I_2 + I_3 \}
\end{gathered}
$$

with

$$
\begin{aligned}
I_1 &= \int_{b_0}^{+\infty} \int_{0}^{F(b_0)} F^{N-2} dF f\left(z_N\right) dz_N = (N-1)^{-1} F\left(b_0\right)^{N-1} \int_{b_0}^{+\infty} f\left(z_N\right) dz_N \\
&= (N-1)^{-1} F\left(b_0\right)^{N-1} \left[1 - F\left(b_0\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_{b_0}^{b_0+\Delta} \int_{F(b_0)}^{F(z_N)} F^{N-2} dF f\left(z_{_N}\right) dz_{_N} \\
&= (N-1)^{-1} \int_{b_0}^{b_0+\Delta} \left[F^{N-1}\left(z_N\right) - F^{N-1}\left(b_0\right)\right] f\left(z_{_N}\right) dz_{_N} \\
&= (N-1)^{-1} \left\{ \int_{F(b_0)}^{F(b_0+\Delta)} F^{N-1} dF - F^{N-1}\left(b_0\right)\left[F\left(b_0+\Delta\right) - F\left(b_0\right)\right] \right\} \\
&= (N-1)^{-1} \left\{ N^{-1}\left[F^N\left(b_0+\Delta\right) - F^N\left(b_0\right)\right] - F^{N-1}\left(b_0\right)\left[F\left(b_0+\Delta\right) - F\left(b_0\right)\right] \right\}
\end{aligned}
$$

$$
I_3 = (N-1)^{-1} \left[F\left(b_0+\Delta\right)^{N-1} - F\left(b_0\right)^{N-1}\right]\left[1 - F\left(b_0+\Delta\right)\right]
$$